

A NUMERICAL SOLUTION TO THE CATTANEO-MINDLIN PROBLEM PART I: ALGORITHM DESCRIPTION

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Abstract: The Cattaneo-Mindlin problem addresses the elastic spherical contact undergoing normal and tangential loading, when the tangential force is less than a limiting value inducing a gross-slip regime. The principles of problem digitization and the state of the art in solving the normal contact problem are overviewed. A numerical model for the Cattaneo-Mindlin problem is advanced and solved via conjugate gradient method. Shear tractions are derived as the solution of the linear system arising from discretization of integral equation of deformation. Complementarity conditions, assessing the stick or slip status of cells in the contact area, and static force equilibrium, are enforced during conjugate gradient iterations, thus eliminating the need for additional iterative levels.

1. INTRODUCTION

As analytical models lack the mathematical support for solving the complex equations arising in Contact Mechanics, numerical methods have found great applicability due to their ability in simulating various scenarios of contact geometry or material response. The search for refined algorithms, capable of solving fine meshes and complex patterns of material behavior with a moderate computational effort, remains one of the major challenges to be met.

The problem of elastic contact undergoing normal loading is usually solved under the assumption that bounding surfaces are frictionless or cannot sustain shear tractions, [10]. A numerical model to predict fretting wear under gross slip and partial slip conditions was advanced by Gallego and Nélías, [5]. The point contact of dissimilar materials considering tangential tractions was also investigated numerically by Chen and Wang, [4]. Their model was later applied by Wang, Wang, Wang, Zhu, and Hu, [12], to three-dimensional contact involving elastic layered half-space.

A numerical model for the elastic contact with slip and stick, solved independently by Cattaneo, [2], and by Mindlin, [9], for spherical contact geometry, is advanced in this paper. The approach relies and on the similarity between the models for the normal and for the tangential contact problems considered individually, and on the solution for the elastic normal contact advanced by Polonsky and Keer, [10].

2. OVERVIEW OF NORMAL CONTACT PROBLEM SOLUTION

As numerical resolution of normal contact problem has reached maturity, only a basic description of the algorithm is provided. The framework for the elastic normal contact solver consists of the following assumptions / limitations:

1. Contact area is small compared to dimensions of the contacting bodies; consequently, the half-space approximation holds.
2. Only small strains and small displacements are considered.
3. The contact is dry (not lubricated), and friction is not accounted for.

Numerical solution of elastic contact problem relies on considering continuous distributions as piecewise constant on the elements of a rectangular mesh established in a small surface domain D , expected to include the contact area. Individual patches, of area Δ , are identified by a pair of indices (i, j) , with $1 \leq j \leq N_x$, $1 \leq i \leq N_y$, and $N = N_x N_y$. The

nodal value of any continuous distribution, $f(x(j), y(i))$ over D is denoted by $f(i, j)$. This approach allows transforming the integral contact equation, for which analytical solutions exist only in a few cases, in a linear system of equations in pressure. For the half-space approximation to remain valid, the slope of composite geometry should be small all over D .

Kalker and van Randen, [7], formulated the elastic contact problem with normal loading as a problem of minimization, where the unknown contact area A_c and pressure distribution p , minimize the total complementary energy, under the restrictions that pressure is positive everywhere on the contact area and there is no interpenetration:

$$h(i, j) = h_i(i, j) + w(i, j) - \omega, (i, j) \in D; \quad (1)$$

$$h(i, j) = 0, p(i, j) > 0, (i, j) \in A_c; \quad (2)$$

$$h(i, j) > 0, p(i, j) = 0, (i, j) \in D - A_c; \quad (3)$$

$$\Delta \sum_{(i,j) \in A_c} p(i, j) = W, \quad (4)$$

where h_i denotes the initial contact geometry, h is the gap between the deformed contact surfaces, ω is the rigid-body approach, and W the normal load transmitted through contact.

The model (1)-(4) can be solved efficiently using a modified conjugate gradient method (CGM) originally proposed by Polonsky and Keer, [10]. This algorithm has two main advantages. Firstly, convergence is assured, as mathematical proof of convergence for the CGM was advanced, [6]. Secondly, the algorithm allows for imposing additional restrictions during pressure iterations. Consequently, the a priori unknown contact area is also iterated during pressure correction, by enforcing the complementarity conditions, Eqs. (2) and (3). The force balance condition in Eq. (4) is also imposed while solving the linear system in pressure. This approach excludes the need for additional nested loops, present in earlier contact solvers.

The model for simulating the normal contact was refined by Spinu and Diaconescu, [11], allowing for an eccentric loading which causes the tilting of the common plane of contact.

3. PROBLEM FORMULATION

Cattaneo, [2], and Mindlin, [9], proved that, if a tangential effort T smaller than the one inducing a gross-slip regime, $T_{\text{lim}} = \mu W$, with μ the friction coefficient, is applied in addition to a constant normal loading in a contact with Hertz geometry, the solution of the adherent contact exhibits a singularity at contact periphery, leading to infinite shear tractions. This solution verifies neither Coulomb law, nor Linear Elasticity equations. Consequently, the considered problem cannot be solved in the frame of Linear Elasticity unless a partial slip region is presumed. Therefore, any region in the contact area should be in either slip (also referred to as *micro-slip*) or stick regime. On the stick area A_s , where contact is adherent, a static friction regime is established, with vanishing relative slip distances s between corresponding points on the limiting surfaces. In the slip area, a kinetic friction regime occurs. In both regions, shear tractions q must obey Coulomb's law of friction, relating shear to normal tractions through frictional coefficient μ .

In order to obey the formalism of the Cattaneo-Mindlin problem, it is assumed that the tangential loading consists of a tangential force T acting along direction of $\bar{x}$, and the

effects in the $\bar{x}$ and $\bar{y}$ directions are not coupled. The condition of deformation on the direction of $\bar{y}$ is not verified, and contributions of tractions acting along $\bar{y}$ on the displacements along $\bar{x}$ and $\bar{z}$ are also neglected.

Application of tangential force T leads to tangential displacements u and to rigid-body tangential translation δ . The following discrete model for the Cattaneo-Mindlin problem can be formulated:

$$s(i, j) = u(i, j) - \delta, (i, j) \in A_C; \quad (5)$$

$$|q(i, j)| \leq \mu p(i, j), s(i, j) = 0, (i, j) \in A_S; \quad (6)$$

$$|q(i, j)| = \mu p^{(ij)}, |s(i, j)| > 0, (i, j) \in A_C - A_S; \quad (7)$$

$$\Delta \sum_{(i, j) \in A_C} q(i, j) = T. \quad (8)$$

4. ALGORITHM DESCRIPTION

At this point, solution of the normal contact problem is assumed to be known from the algorithm described in [10]. Consequently, contact area A_C and pressure distribution p are known. The same grid is used in the tangential problem.

Stick area A_S and distribution of shear tractions q are the unknowns of the Cattaneo-Mindlin problem. A comparison of models (5)-(8) and (1)-(4) suggests that the same approach should be used to solve either tangential or normal contact problem. Indeed, Eq. (5) reduces to a linear system of equations having the nodal shear tractions q as unknowns, which can be solved on the stick area A_S , where the relative slip distances s vanish. The system matrix, namely the influence coefficients matrix, is symmetrical and positive definite, therefore conjugate gradient method can also be applied. The stick area, namely the size of the system to be solved, is also a priori unknown. An iterative approach is required to solve the model, and the strong point of CGM is that allows additional restrictions (Eqs. (6)-(8)) to be imposed during residual minimization. With this approach, no additional iterative levels, needed to adjust the system size, or to impose the static equilibrium, are required, resulting in increased computational efficiency.

When computing its elastic deformation, each body is regarded as an elastic half-space. Therefore, tangential displacements u and w in Eq. (5) can be assessed by applying superposition principle to the Boussinesq, [1], and Cerruti, [3], fundamental solutions (or Green functions). The influence coefficients based approach proves its versatility when combined with spectral methods [8] to accelerate the computationally intensive multi-summation process.

Assessment of elastic displacement due to contact tractions (pressure p or shear tractions q) requires coordinate system transformation from the local ones, related to each body, in which initial contact geometry is defined, to the global system. If we assume that the latter matches that of body (2), relative displacement can be expressed as:

$$\begin{cases} u_i = u_i^{(2)} - u_i^{(1)}; \\ w = w^{(2)} + w^{(1)}. \end{cases} \quad (9)$$

On the other hand, static force equilibrium requires that:

$$\begin{cases} p^{(2)} = p^{(1)} = p; \\ q^{(2)} = -q^{(1)} = q. \end{cases} \quad (10)$$

By plugging Eq. (10) into relation (9), one can obtain:

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} K_{xx}^{(2)} + K_{xx}^{(1)} & K_{xz}^{(2)} - K_{xz}^{(1)} \\ K_{zx}^{(2)} - K_{zx}^{(1)} & K_{zz}^{(2)} + K_{zz}^{(1)} \end{bmatrix} \otimes \begin{bmatrix} q \\ p \end{bmatrix}, \quad (11)$$

where symbol “ $\otimes$ ” is used to denote discrete cyclic convolution and K_{ij} is the influence coefficients matrix expressing displacements in the direction of $\vec{i}$ due to a unit traction acting along $\vec{j}$. Computation of matrix K is detailed in Appendix. Eq. (11) suggests that, in the general case, normal and tangential effects are connected, as shear tractions are needed to compute the normal displacement, and contact pressure is required in tangential displacement assessment.

The newly proposed algorithm for solving numerically the elastic contact problem under partial slip regime can be summarized in the steps described in the following paragraphs.

Firstly, the initial guess for shear tractions is computed from static force equilibrium, Eq. (8), assuming all cells in the contact area are in stick regime. The mean value of shear tractions over stick area makes a good first approximation, allowing inception of iterative process. Auxiliary variables are initialized: $\theta = 0$, $S_0 = 1$, as well as the descent direction, the latter with vanishing values over A_C .

The following operations are looped until the imposed tolerance is reached.

1. Compute tangential displacement field over D using Eq. (11).
2. Estimate the rigid-body tangential translation δ . To this end, Eq. (5) is plugged into relation (6), resulting in a linear system of equations having a number of equations equal to the number of cells in the stick region. When convergence is reached, these equations are not all independent. However, during iterations, they appear to be independent. Therefore, to get the best estimate of δ , the system is treated as over-determined, and the best fit is sought:

$$\delta = \frac{\sum_{(i,j) \in A_s} u(i,j)}{\sum_{(i,j) \in A_s} 1}. \quad (12)$$

3. Start the conjugate gradient loop to solve Eq. (5) on the stick area. To this end, compute the relative slip distances, which represent the residual in terms of CGM formulation:

$$s(i,j) = u(i,j) - \delta, (i,j) \in A_C. \quad (13)$$

4. Compute the square sum of the residual on the stick area:

$$S = \sum_{(i,j) \in A_s} s^2(i,j). \quad (14)$$

5. Compute the descent direction d in the CGM algorithm. In the multidimensional space of shear tractions $q^{(ij)}, (i, j) \in A_S$, every new descent direction is constructed from the residual, so that it is K -orthogonal to all previous residuals and search directions:

$$d(i, j) = \begin{cases} s(i, j) + d(i, j)S\theta/S_0, & (i, j) \in A_S; \\ 0, & (i, j) \in A_C - A_S. \end{cases} \quad (15)$$

6. Overwrite the old value of the square sum of the residual with the newly computed one: $S_0 = S$.

7. Estimate the convolution over D between influence coefficients matrix and descent direction:

$$r = (K_{xx}^{(2)} + K_{xx}^{(1)}) \otimes d. \quad (16)$$

8. Use this result to assess the length of the step to be made along the descent direction:

$$\alpha = s(i, j) \cdot d(i, j) / [r(i, j) \cdot d(i, j)], (i, j) \in A_S. \quad (17)$$

9. Memorize the current solution in a new variable in order to compute the relative error between consecutive iterations:

$$q_0(i, j) = q(i, j), (i, j) \in A_C. \quad (18)$$

10. Update the system solution by making a step of length α along the direction of d :

$$q(i, j) \leftarrow q(i, j) - \alpha \cdot d(i, j), (i, j) \in A_S. \quad (19)$$

11. Verify complementarity conditions in Eqs. (6) and (7) to determine the stick or slip status of every cell in the contact area. Cells for which Coulomb friction law is not verified are removed from the stick region, and the corresponding shear tractions are set to the value of limiting friction (or maximum static friction). In the same time, cells with micro-slip vectors $s(i, j)$ not opposite to corresponding shear tractions are removed from the slip region and included in A_S .

$$q(i, j) \leftarrow \mu p(i, j) \cdot \text{sign}(q(i, j)), (i, j) \in \{(i, j) : |q(i, j)| > \mu p(i, j)\}, \quad (20)$$

$$A_S \leftarrow A_S - \{(i, j) : |q(i, j)| > \mu p(i, j)\} \cup \{(i, j) : q(i, j)s(i, j) > 0\}. \quad (21)$$

If any cell is removed or reenters the stick area, auxiliary variable θ is set to zero, otherwise it is set to unity. This variable allows resetting the conjugate directions once new cells enter or leave the system. Indeed, these new cells have no precedent in the minimization process and therefore a new search must be conducted.

12. Impose static equilibrium by adjusting the current solution according to the following relation:

$$q(i, j) \leftarrow q(i, j) + a, (i, j) \in A_S, \quad (22)$$

where the unknown term a is computed by plugging Eq. (22) into Eq. (8):

$$a = \left[T/\Delta - \sum_{(i,j) \in A_c} q(i,j) \right] / \sum_{(i,j) \in A_s} 1 \quad (23)$$

13. Verify convergence criterion:

$$\sum_{(i,j) \in A_c} |q(i,j) - q_0(i,j)| \leq \varepsilon. \quad (24)$$

The flowchart of the newly proposed algorithm is presented in Fig. 1.

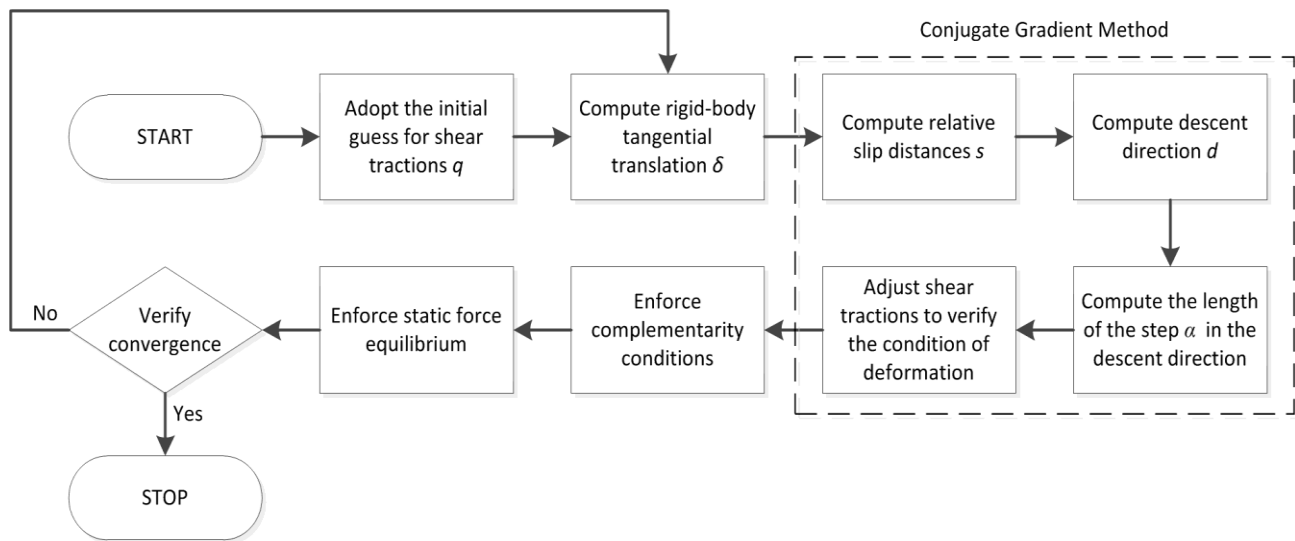


Figure 1. Algorithm flowchart for the Cattaneo-Mindlin problem

5. CONCLUSIONS

This paper advances a fast and robust numerical solver for the Cattaneo-Mindlin problem, namely the problem of elastic contact undergoing normal and tangential loading, when a slip-stick regime is established.

The state of the art in normal contact modeling is overviewed, as well as the framework for the contact with tangential loading and friction. The equations related to the tangential direction are: integral condition of deformation, written in the frame of half-space theory, complementarity conditions, assessing the slip or stick status of every cell in the contact area, and static force equilibrium.

The shear tractions are computed by solving via conjugate gradient method the linear system resulting from digitization of integral condition of deformation along the tangential direction. Complementarity conditions and the static force equilibrium are imposed during iteration of shear tractions. The stick and slip regions, as well as the rigid-body tangential translation, are determined in the course of conjugate gradient iterations.

The newly proposed algorithm is expected to solve the problem of slip-stick elastic contact for any initial contact geometry in the framework of the half-space theory.

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APPENDIX

Elastic displacements u_{ij} along direction of $\vec{i}$ due to a unit concentrated force acting in origin along direction of $\vec{j}$ were derived by Boussinesq, [1], if $j = z$, and by Cerruti, [3], for $j = x$:

$$u_{xx}(x, y, z) = \frac{1}{4\pi\mu} \left\{ \frac{1}{r} + \frac{x^2}{r^3} + \frac{1-2\nu}{r+z} \left[1 - \frac{x^2}{r(r+z)} \right] \right\}; \quad (25)$$

$$u_{zz}(x, y, z) = \frac{1}{4\pi\mu} \left[\frac{z^2}{r^3} + 2(1-\nu) \frac{1}{r} \right]; \quad (26)$$

$$u_{xz}(x, y, z) = u_{zx}(x, y, z) = \frac{1}{4\pi\mu} \left[\frac{xz}{r^3} - (1-2\nu) \frac{x}{r(r+z)} \right]; \quad (27)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

If the steps of the grid in the corresponding directions are denoted by Δ_x and Δ_y , with $\Delta = \Delta_x \Delta_y$, then the influence coefficients matrix in Eqs. (11) and (16) can be computed using the following relation:

$$K_{\ell m}^{(n)}(i, j) = k_{\ell m}^{(n)}(x(j) + \Delta_x/2, y(i) + \Delta_y/2) + k_{\ell m}^{(n)}(x(j) - \Delta_x/2, y(i) - \Delta_y/2) \dots \\ - k_{\ell m}^{(n)}(x(j) + \Delta_x/2, y(i) - \Delta_y/2) - k_{\ell m}^{(n)}(x(j) - \Delta_x/2, y(i) + \Delta_y/2), \quad (28)$$

with functions $k_{ij}^{(n)}$ defined as primitives of functions (25) - (27):

$$k_{xx}^{(n)}(x, y) = \frac{1 + \nu^{(n)}}{\pi E^{(n)}} \left[y \ln(x + \sqrt{x^2 + y^2}) - y + (1 - \nu^{(n)}) x \ln(y + \sqrt{x^2 + y^2}) \right]; \quad (29)$$

$$k_{zz}^{(n)}(x, y) = \frac{1 - (\nu^{(n)})^2}{\pi E^{(n)}} \left[x \ln(x_2 + \sqrt{x^2 + y^2}) + y \ln(x_1 + \sqrt{x^2 + y^2}) \right]; \quad (30)$$

$$k_{xz}^{(n)}(x, y) = k_{zx}^{(n)}(x, y) = -\frac{(1 + \nu^{(n)})(1 - 2\nu^{(n)})}{2\pi E^{(n)}} \left[2x \tan^{-1}\left(\frac{\sqrt{x^2 + y^2} - y}{x}\right) - y \ln(\sqrt{x^2 + y^2}) \right], \quad (31)$$

where $E^{(n)}$ and $\nu^{(n)}$ are the Young modulus and the Poisson's ratio of the body (n) , $n = 1, 2$.